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**A Technique for Determining Particle Anisotropies
using the Voyager Low Energy Telescopes**

by

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Abstract

In regions of space where there is no apriori information about the orientation of possible particle anisotropies, the four Voyager Low Energy Telescopes (LETs) can be used to specify the direction and magnitude of any first-order anisotropy. If directions in space can be identified that represent symmetry axes for particle motion (e.g. magnetic field lines or an observer's direction of motion with respect to a particle population), the LETs can be used to determine the magnitudes of higher-order, as well as first-order, anisotropies about these directions. The coefficients of the linear equations that relate the anisotropy magnitudes to the telescope counting rates are calculated in this report.

The counting rate of a particle telescope in a flux $F(\vartheta, \varphi, E)$ is

$$C = \int_{E_1}^{E_2} dE \int_{\Omega} d\Omega F(\vartheta, \varphi, E) A(\vartheta, \varphi) \quad (1)$$

where $E_1 - E_2$ is the energy range of particles whose energy loss in the detectors exceeds the counting thresholds and $A(\vartheta, \varphi)$ is the effective area of the telescope (e.g. the overlap area of two detectors). For an isotropic flux, $j(E)$,

$$C = \int_{E_1}^{E_2} dE j(E) \int_{\Omega} d\Omega A(\vartheta, \varphi) = A\Omega \int_{E_1}^{E_2} dE j(E)$$

where $A\Omega \equiv \int_{\Omega} d\Omega A(\vartheta, \varphi)$ is the telescope geometry factor. For an anisotropic flux, one would like to use the counting rates in two or more independent telescopes to obtain information about the angular dependence of $F(\vartheta, \varphi, E)$ or, assuming F is separable as $F(\vartheta, \varphi, E) = I(\vartheta, \varphi) j(E)$, about $I(\vartheta, \varphi)$. The specific application considered in this report is the four LETs that are part of the Cosmic Ray Subsystem on the Voyager spacecraft. The approach here will be to expand $I(\vartheta, \varphi)$ in a series of spherical harmonics and do the angular integration in equation 1 numerically for each term of the series using the appropriate formula for $A(\vartheta, \varphi)$. The resulting linear equations can then be solved to give the coefficients in the spherical harmonic expansion in terms of the four counting rates.

Since the spherical harmonics $Y_{lm}(\vartheta, \varphi)$ are a complete set of functions, any function $I(\vartheta, \varphi)$ (with sufficient continuity properties) can be expressed as a series in terms of them,

$$I(\vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\vartheta, \varphi)$$

where the a_{lm} are constants. The LETs give four independent equations

that can be used to determine four of the constants. Given no additional information, we chose to determine the first four a_{lm} . The expansion is most convenient in terms of the spherical harmonics defined by Morse and Feshbach (1953, p.1271):

$$I(\vartheta, \varphi) = a_{00} + a_{10}\cos\vartheta + a_{11}\sin\vartheta\cos\varphi + a_{1-1}\sin\vartheta\sin\varphi.$$

This equation can be put into a simpler form by rotating the coordinate system by the Euler angles (Goldstein, 1950, p. 107) ϑ_{e1} , φ_{e1} , and ψ_{e1} such that

$$\begin{aligned} \cos\vartheta_{e1} &= \frac{a_{10}}{\sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}} & \sin\vartheta_{e1} &= \frac{\sqrt{a_{11}^2 + a_{1-1}^2}}{\sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}} \\ \cos\varphi_{e1} &= -\frac{a_{1-1}}{\sqrt{a_{11}^2 + a_{1-1}^2}} & \sin\varphi_{e1} &= \frac{a_{11}}{\sqrt{a_{11}^2 + a_{1-1}^2}} \\ \psi_{e1} &= \text{arbitrary} & (\text{choose } = 0). \end{aligned}$$

The result is

$$I(\vartheta, \varphi) = a_0 + a_1 \cos\vartheta_1 \quad (2)$$

where $a_0 = a_{00}$, $a_1 = \sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}$ and ϑ_1 is the angle relative to the new ***z***-axis. This form shows explicitly that the LETs can measure the magnitude and direction of a Compton-Getting-type anisotropy. The four parameters determined in this case are a_0 , a_1 , ϑ_{e1} , and φ_{e1} .

In some environments, reasonable guesses can be made for the values of one or more of these constants and the LETs can then be used to specify the parameters of higher order spherical harmonics. For instance, in a magnetosphere one might assume that the only first-order anisotropy is due to the corotation of the particle distribution past the observer. In this case, the direction of the axis of the anisotropy (ϑ_{e1} and φ_{e1}) is known, and the extra two degrees of freedom can be used to determine $\cos^2\vartheta$ terms that pertain to pitch-angle distributions about the field lines.

If it is assumed that each higher order spherical harmonic term is axially symmetric about some axis, $\hat{\mathbf{e}}_i$, it can be shown (Sanderson and

Hynds, 1977, and references therein) that $I(\vartheta, \varphi)$ can be represented as:

$$I(\vartheta, \varphi) = a_0 + a_1 \cos \vartheta_1 + a_2 \frac{3}{2} (\cos^2 \vartheta_2 - \frac{1}{3}) + a_3 \frac{5}{2} (\cos^3 \vartheta_3 - \frac{3}{5} \cos \vartheta_3) + \dots \quad (3)$$

where ϑ_1 , ϑ_2 and ϑ_3 are measured with respect to \hat{e}_1 , \hat{e}_2 , and \hat{e}_3 . There are clearly fewer parameters in the $\cos^2 \vartheta$ and $\cos^3 \vartheta$ terms in this equation than in the same order terms in the spherical harmonic equation. For the $\cos^2 \vartheta$ term, there are five spherical harmonic coefficients a_{20} , a_{21} , a_{2-1} , a_{22} and a_{2-2} , but in equation 3 only three parameters appear; namely, a_2 and the Euler angles ϑ_{e2} and φ_{e2} of the \hat{e}_2 axis. Similarly, for the a_3 term, the seven spherical harmonic coefficients correspond to the three parameters a_3 , ϑ_{e3} , and φ_{e3} above. The reason is that, to be written in the form of equation 3, the a_{2m} and a_{3m} terms in the spherical harmonic expansions were assumed to be axially symmetric about some axes \hat{e}_2 and \hat{e}_3 . The Maxwell theory of poles of the spherical harmonics (see e.g. Sanderson and Hynds, 1977), from which equation 3 was derived, requires, in general, two axes \hat{e}_{2a} and \hat{e}_{2b} to represent the a_{2m} spherical harmonic terms. There are therefore five parameters, a_2 , ϑ_{e2a} , φ_{e2a} , ϑ_{e2b} and φ_{e2b} to match the five a_{2m} . For the a_{3m} terms, three axes (six Euler angles) are required. The axial symmetry assumption means that two a_2 axes line up and that the three a_3 axes line up. The expansion for $I(\vartheta, \varphi)$ given in equation 3 should therefore only be used if there is some physical basis for assuming axially symmetric $\cos^2 \vartheta$ or $\cos^3 \vartheta$ distributions. In the previous magnetosphere example, the field lines impose a natural symmetry condition on the pitch angle distributions.

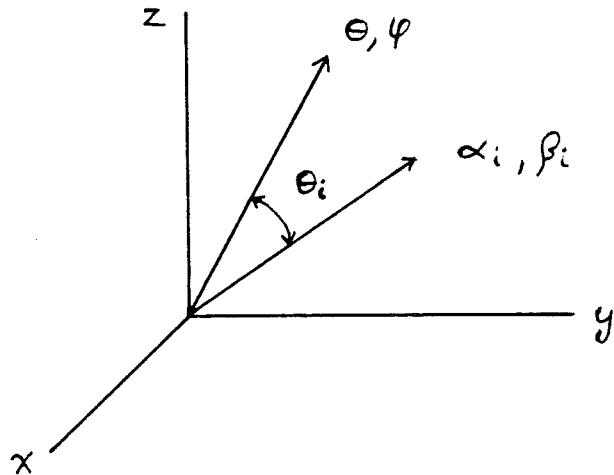
The next step is to perform the angular integration in equation 1. For $F(E, \vartheta, \varphi) = j(E) I(\vartheta, \varphi)$,

$$\begin{aligned} C = & \int_{E_1}^{E_2} dE j(E) \int_0^{2\pi} d\varphi \int_0^\pi \sin \vartheta d\vartheta A(\vartheta, \varphi) [a_0 + a_1 \cos \vartheta_1 \\ & + a_2 \frac{3}{2} (\cos^2 \vartheta_2 - \frac{1}{3}) + a_3 \frac{5}{2} (\cos^3 \vartheta_3 - \frac{3}{5} \cos \vartheta_3)]. \end{aligned} \quad (4)$$

In order to actually do the integrals ϑ_i must be transformed into ϑ and φ using

$$\cos\vartheta_i = \cos\vartheta \cos\alpha_i + \sin\vartheta \sin\alpha_i \cos(\varphi - \beta_i) \quad i = 1, 2, 3, \dots \quad (5)$$

where α_i and β_i are the angles of the i^{th} symmetry axis (\hat{e}_i) in the telescope coordinate frame (z axis = telescope symmetry axis).



It is shown in the Appendix that, as one would expect from the cylindrical symmetry, the integral does not depend on β_i . We also need an explicit expression for $A(\vartheta, \varphi)$. For the LET case the area-overlap formula for two separated circular disks can be used:

$$\begin{aligned} A(\vartheta, \varphi) &= \pi r_s^2 \cos\vartheta & 0 \leq \vartheta < \vartheta_c \\ &= \cos\vartheta \left\{ \frac{r_1^2}{2} [2\Phi_1 - \sin 2\Phi_1] + \frac{r_2^2}{2} [2\Phi_2 - \sin 2\Phi_2] \right\} & \vartheta_c \leq \vartheta < \vartheta_m \\ &= 0 & \vartheta_m \leq \vartheta \end{aligned}$$

(Sullivan, 1971) where:

r_1 = radius of detector 1

r_2 = radius of detector 2

x_s = smaller of r_1 and r_2

$$\vartheta_c = \tan^{-1} \frac{|r_1 - r_2|}{d}$$

$$\vartheta_m = \tan^{-1} \frac{|r_1 + r_2|}{d}$$

$$\Phi_1 = \cos^{-1} \left[\frac{r_1^2 + d^2 \tan^2 \vartheta - r_2^2}{2dr_1 \tan \vartheta} \right]$$

$$\Phi_2 = \cos^{-1} \left[\frac{r_2^2 + d^2 \tan^2 \vartheta - r_1^2}{2dr_2 \tan \vartheta} \right]$$

d = detector separation

Note that due to the assumed cylindrical symmetry, $A(\vartheta, \varphi)$ does not depend on φ .

The integrals can now be done. A convenient way to express the answer is to define

$$K_0 = 2\pi \int_0^\pi d\vartheta A(\vartheta) \sin \vartheta = A\Omega$$

$$K_1(\alpha_1) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta_1 \sin \vartheta \quad (7)$$

$$K_2(\alpha_2) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta A(\vartheta) \frac{3}{2} \left[\cos^2 \vartheta_2 - \frac{1}{3} \right] \sin \vartheta$$

$$K_3(\alpha_3) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta A(\vartheta) \frac{5}{2} \left[\cos^3 \vartheta_3 - \frac{3}{5} \cos \vartheta_3 \right] \sin \vartheta$$

Comparing equations 4 and 7, we see that the counting rate in a telescope can now be succinctly expressed as

$$C = \int_{E_1}^{E_2} dE j(E) \sum_{i=0}^n a_i K_i(\alpha_i) . \quad (8)$$

In the case of the LETs there are four such equations so that in principle a_0 , a_1 , a_2 , and a_3 can be determined. In the Appendix, the values of K_0 , K_1 , K_2 , and K_3 are calculated for both a window-L1 and a L1-L2 configuration. The results are listed in the computer output at the end of this report. As is indicated in equation 8 and shown in the Appendix, the values of K_i depend on the angles α_i between the telescope axis and the i^{th} symmetry axis. Fortunately, however, the α_i dependence comes out of the integrals, so that the integrals only need be done once for a given detector configuration. For example, using the numbers listed in the computer output, the fully expanded expression for the counting rate for the nominal L1-L2 configuration ($r_1=r_2=0.95$ cm, $d=4.08$ cm) is

$$C = \int_{E_1}^{E_2} dE j(E) [a_0 \cdot 0.4367 + a_1 \cdot 0.4263 \cdot \cos\alpha_1 + \\ a_2 (0.6091 \cdot \cos^2\alpha_2 - 0.2030) + a_3 (0.9432 \cdot \cos^3\alpha_3 - 0.5659 \cdot \cos\alpha_3)]$$

Appendix

In this section the equations for K_i (equations 7) are put into forms that can be numerically integrated given the detector configuration. First, since $A(\vartheta)$ does not depend on φ , all integrals involving φ can be done analytically. A useful equation is

$$\int_0^{2\pi} d\varphi \cos^n(\varphi - \beta_i) = \int_0^{2\pi} d\varphi \cos^n \varphi = \begin{cases} 2\pi & n = 0 \\ 0 & n \text{ odd} \\ \pi & n = 2, 4, 6, \dots \end{cases}$$

As expected from the cylindrical symmetry, the β_i dependence drops out so that β_i can be set to zero in equation 5.

K_0 :

$$K_0 = 2\pi \int_0^{\pi} d\vartheta A(\vartheta) \sin \vartheta$$

$K_1(\alpha_1)$:

$$\cos \vartheta_1 = \cos \vartheta \cos \alpha_1 + \cos \varphi \text{ terms}$$

$$K_1(\alpha_1) = \cos \alpha_1 2\pi \int_0^{\pi} d\vartheta A(\vartheta) \cos \vartheta \sin \vartheta$$

$K_2(\alpha_2)$:

$$\begin{aligned} \cos^2 \vartheta_2 &= \cos^2 \vartheta \cos^2 \alpha_2 + \sin^2 \vartheta \sin^2 \alpha_2 \cos^2 \varphi + \cos \varphi \text{ terms} \\ &= \cos^2 \alpha_2 (\cos^2 \vartheta - \sin^2 \vartheta \cos^2 \varphi) + \sin^2 \vartheta \cos^2 \varphi + \cos \varphi \text{ terms} \end{aligned}$$

$$K_2(\alpha_2) = \frac{3}{2} \cos^2 \alpha_2 \left[2\pi \int_0^{\pi} d\vartheta A(\vartheta) \cos^2 \vartheta \sin \vartheta - \int_0^{2\pi} d\varphi \cos^2 \varphi \int_0^{2\pi} d\vartheta A(\vartheta) \sin^2 \vartheta \sin \vartheta \right]$$

$$+ \frac{3}{2} \int_0^{2\pi} d\varphi \cos^2 \varphi \int_0^{2\pi} d\vartheta A(\vartheta) \sin^2 \vartheta \sin \vartheta - \frac{1}{2} 2\pi \int_0^{\pi} d\vartheta A(\vartheta) \sin \vartheta$$

$$\begin{aligned}
 &= \frac{3\pi}{2} \cos^2 \alpha_2 \left[2 \int_0^\pi d\vartheta A(\vartheta) \cos^2 \vartheta \sin \vartheta - \int_0^\pi d\vartheta A(\vartheta) \sin^3 \vartheta \right] \\
 &\quad + \pi \left[\frac{3}{2} \int_0^\pi d\vartheta A(\vartheta) \sin^3 \vartheta - \int_0^\pi d\vartheta A(\vartheta) \sin \vartheta \right]
 \end{aligned}$$

$K_3(\alpha_3)$:

$$\begin{aligned}
 \cos^3 \vartheta_3 &= \cos^3 \vartheta \cos^3 \alpha_3 + 3 \cos \vartheta \cos \alpha_3 \sin^2 \vartheta \sin^2 \alpha_3 \cos^2 \varphi + \text{odd powers of } \cos \varphi \\
 &= \cos^3 \alpha_3 (\cos^3 \vartheta - 3 \cos \vartheta \sin^2 \vartheta \cos^2 \varphi) + \\
 &\quad \cos \alpha_3 3 \cos \vartheta \sin^2 \vartheta \cos^2 \varphi + \text{odd powers of } \cos \varphi
 \end{aligned}$$

$$\begin{aligned}
 K_3(\alpha_3) &= \frac{5}{2} \cos^3 \alpha_3 \left[2\pi \int_0^\pi d\vartheta A(\vartheta) \cos^3 \vartheta \sin \vartheta - 3 \int_0^{2\pi} d\varphi \cos^2 \varphi \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin^3 \vartheta \right] \\
 &\quad + \cos \alpha_3 \left[\frac{3 \cdot 5}{2} \int_0^{2\pi} d\varphi \cos^2 \varphi \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin^3 \vartheta - 2\pi \frac{3}{2} \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin \vartheta \right] \\
 &= \frac{5}{2} \pi \cos^3 \alpha_3 \left[2 \int_0^\pi d\vartheta A(\vartheta) \cos^3 \vartheta \sin \vartheta - 3 \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin^3 \vartheta \right] \\
 &\quad - \pi \cos \alpha_3 \left[\frac{15}{2} \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin^3 \vartheta - 3 \int_0^\pi d\vartheta A(\vartheta) \cos \vartheta \sin \vartheta \right]
 \end{aligned}$$

The program **ANNIE** performs these integrals numerically for given detector configurations defined by r_1 , r_2 and d (equation 6). An output and a program listing are at the end of this report. Note that K_0 is the geometry factor of the telescope and can be analytically calculated (Sullivan, 1968). The values of K_0 returned by the program are consistent with such calculations. The test case listed on the fourth page of the output is for two detectors separated by a large distance compared to their radii, so that $A(\vartheta)$ becomes essentially a delta function centered at

$\vartheta=0$. Considering the above equations, one can show that the program gives the expected result:

$$C = \int_{E_1}^{E_2} dE j(E) \cdot A\Omega \cdot [a_0 + a_1 \cos\alpha_1 + \\ a_2 \frac{3}{2}(\cos^2\alpha_2 - \frac{1}{3}) + a_3 \frac{5}{2}(\cos^3\alpha_3 - \frac{3}{5}\cos\alpha_3)]$$

where $A\Omega=K_0=9.87 \times 10^{-6}$. Note the similarity in form between this equation and equation 3.

References

- Goldstein, H., *Classical Mechanics*, Addison-Wesley, Massachusetts, 1950.
- Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, part II, McGraw-Hill, New York, 1953.
- Sanderson, T. R. and R. J. Hynds, Multiple telescope measurements of particle anisotropies in space, *Planet. Space Sci.*, 25, 799, 1977.
- Sullivan, J. D., Geometrical factor and directional response of single and multi-element particle telescopes, *Nuclear Instr. and Methods*, 95, 5, 1971.

-/-

NOMINAL L1-L2

R1 = 0.9500 R2 = 0.9500 D = 4.080
THETAC = 0.0 THETAM = 0.436 DELTH = 0.0020

INTEGRAL(A SIN)=0.06950 INTEGRAL(A COS SIN)=0.06785
INTEGRAL(A COS**2 SIN)=0.06626 INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06473 INTEGRAL(A COS SIN**3)=0.00312

K0 = A OMEGA = 0.4367

K1 = COS(ALPHA1) * 0.4263

K2 = COS**2(ALPHA2) * 0.6091 + -0.2030

K3 = COS**3(ALPHA3) * 0.9432 + COS(ALPHA3) * -0.5659

NOMINAL WINDOW-L1

R1 = 1.7400 R2 = 0.9500 D = 1.550
THETAC = 0.471 THETAM = 1.048 DELTH = 0.0020

INTEGRAL(A SIN)=0.73153 INTEGRAL(A COS SIN)=0.61663
INTEGRAL(A COS**2 SIN)=0.52847 INTEGRAL(A SIN**3)=0.20308
INTEGRAL(A COS**3 SIN)=0.45957 INTEGRAL(A COS SIN**3)=0.15709

K0 = A OMEGA = 4.5964

K1 = COS(ALPHA1) * 3.8744

K2 = COS**2(ALPHA2) * 4.0238 + -1.3412

K3 = COS**3(ALPHA3) * 3.5175 + COS(ALPHA3) * -2.1102

VGR 1 LET A L1-L2

R1 = 0.9505 R2 = 0.9484 D = 4.076
THETAC = 0.001 THETAM = 0.436 DELTH = 0.0020

INTEGRAL(A SIN)=0.06947 INTEGRAL(A COS SIN)=0.06782
INTEGRAL(A COS**2 SIN)=0.06623 INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06470 INTEGRAL(A COS SIN**3)=0.00312

K0 = A OMEGA = 0.4365

K1 = COS(ALPHA1) * 0.4261

K2 = COS**2(ALPHA2) * 0.6088 + -0.2029

K3 = COS**3(ALPHA3) * 0.9427 + COS(ALPHA3) * -0.5656

VGR 1 LET B L1-L2
R1 = 0.9479 R2 = 0.9519 D = 4.081
THETAC = 0.001 THETAM = 0.436 DELTH = 0.0020

INTEGRAL(A SIN)=0.06945 INTEGRAL(A COS SIN)=0.06779
INTEGRAL(A COS**2 SIN)=0.06620 INTEGRAL(A SIN**3)=0.00324
INTEGRAL(A COS**3 SIN)=0.06467 INTEGRAL(A COS SIN**3)=0.00312

K0 = A OMEGA = 0.4363
K1 = COS(ALPHA1) * 0.4259
K2 = COS**2(ALPHA2) * 0.6086 + -0.2029
K3 = COS**3(ALPHA3) * 0.9425 + COS(ALPHA3) * -0.5655

VGR 1 LET C L1-L2
R1 = 0.9487 R2 = 0.9478 D = 4.079
THETAC = 0.000 THETAM = 0.435 DELTH = 0.0020

INTEGRAL(A SIN)=0.06905 INTEGRAL(A COS SIN)=0.06741
INTEGRAL(A COS**2 SIN)=0.06583 INTEGRAL(A SIN**3)=0.00322
INTEGRAL(A COS**3 SIN)=0.06431 INTEGRAL(A COS SIN**3)=0.00309

K0 = A OMEGA = 0.4338
K1 = COS(ALPHA1) * 0.4235
K2 = COS**2(ALPHA2) * 0.6053 + -0.2018
K3 = COS**3(ALPHA3) * 0.9374 + COS(ALPHA3) * -0.5624

VGR 1 LET D L1-L2
R1 = 0.9460 R2 = 0.9466 D = 4.075
THETAC = 0.000 THETAM = 0.435 DELTH = 0.0020

INTEGRAL(A SIN)=0.06863 INTEGRAL(A COS SIN)=0.06700
INTEGRAL(A COS**2 SIN)=0.06544 INTEGRAL(A SIN**3)=0.00319
INTEGRAL(A COS**3 SIN)=0.06393 INTEGRAL(A COS SIN**3)=0.00307

K0 = A OMEGA = 0.4312
K1 = COS(ALPHA1) * 0.4210
K2 = COS**2(ALPHA2) * 0.6017 + -0.2006
K3 = COS**3(ALPHA3) * 0.9320 + COS(ALPHA3) * -0.5592

VGR 2 LET A L1-L2
R1 = 0.9471 R2 = 0.9491 D = 4.075
THETAC = 0.000 THETAM = 0.436 DELTH = 0.0020

INTEGRAL(A SIN)=0.06913 INTEGRAL(A COS SIN)=0.06748
INTEGRAL(A COS**2 SIN)=0.06590 INTEGRAL(A SIN**3)=0.00323
INTEGRAL(A COS**3 SIN)=0.06438 INTEGRAL(A COS SIN**3)=0.00310

K0 = A OMEGA = 0.4344
K1 = COS(ALPHA1) * 0.4240
K2 = COS**2(ALPHA2) * 0.6059 + -0.2020
K3 = COS**3(ALPHA3) * 0.9383 + COS(ALPHA3) * -0.5630

VGR 2 LET B L1-L2
R1 = 0.9451 R2 = 0.9483 D = 4.064
THETAC = 0.001 THETAM = 0.436 DELTH = 0.0020

INTEGRAL(A SIN)=0.06908 INTEGRAL(A COS SIN)=0.06743
INTEGRAL(A COS**2 SIN)=0.06585 INTEGRAL(A SIN**3)=0.00323
INTEGRAL(A COS**3 SIN)=0.06433 INTEGRAL(A COS SIN**3)=0.00310

K0 = A OMEGA = 0.4340
K1 = COS(ALPHA1) * 0.4237
K2 = COS**2(ALPHA2) * 0.6054 + -0.2018
K3 = COS**3(ALPHA3) * 0.9373 + COS(ALPHA3) * -0.5624

VGR 2 LET C L1-L2
R1 = 0.9502 R2 = 0.9424 D = 4.084
THETAC = 0.002 THETAM = 0.434 DELTH = 0.0020

INTEGRAL(A SIN)=0.06835 INTEGRAL(A COS SIN)=0.06674
INTEGRAL(A COS**2 SIN)=0.06519 INTEGRAL(A SIN**3)=0.00317
INTEGRAL(A COS**3 SIN)=0.06369 INTEGRAL(A COS SIN**3)=0.00304

K0 = A OMEGA = 0.4295
K1 = COS(ALPHA1) * 0.4193
K2 = COS**2(ALPHA2) * -0.5994 + -0.1998
K3 = COS**3(ALPHA3) * 0.9287 + COS(ALPHA3) * -0.5572

VGR 2 LET D = L1-L2
R1 = 0.9499 R2 = 0.9447 D = 4.061
THETAC = 0.001 THETAM = 0.437 DELTH = 0.0020

INTEGRAL(A SIN)=0.06934 INTEGRAL(A COS SIN)=0.06768
INTEGRAL(A COS**2 SIN)=0.06609 INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06456 INTEGRAL(A COS SIN**3)=0.00312

K0 = A OMEGA = 0.4357
K1 = COS(ALPHA1) * 0.4252
K2 = COS**2(ALPHA2) * 0.6075 + -0.2025
K3 = COS**3(ALPHA3) * 0.9405 + COS(ALPHA3) * -0.5643

TEST CASE
R1 = 1.000E 00 R2 = 1.000E 00 D = 1.000E 03
THETAC = 0.0 THETAM = 2.000E-03 DELTH = 1.000E-05
INTEGRAL(A SIN)= 1.571E-06 INTEGRAL(A COS SIN)= 1.571E-06
INTEGRAL(A COS**2 SIN)= 1.571E-06 INTEGRAL(A SIN**3)= 1.571E-12
INTEGRAL(A COS**3 SIN)= 1.571E-06 INTEGRAL(A COS SIN**3)= 1.571

K0 = A OMEGA = 5.865E-06
K1 = CCS(ALPHA1) * 5.865E-06
K2 = CCS**2(ALPHA2) * 1.480E-05 + -4.935E-06
K3 = CCS**3(ALPHA3) * 2.467E-05 + COS(ALPHA3) * -1.480E-05

```
//ANNIE      JOB      (98221,CZG,SRL), 'N. GEHRELS', TIME=(2,00)
//          EXEC FORTG
//FORT      DD      *
C -----
C           READ IN DETECTOR RADII AND SEPARATION
C -----
DIMENSION LABEL(19)
COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
5  READ(5,1111,END=999) LABEL
1111 FORMAT(19A4)
      READ(5,1112) R1,R2,D
1112 FORMAT(3F10.0)
C -----
C           INITIALIZE VARIABLES
C -----
CALL INIT(R1,R2,D)
DELTH=0.002
C -----
C           OUTPUT THE A HEADER
C -----
WRITE(6,3333) LABEL
3333 FORMAT(////1X19A4)
      WRITE(6,3334) R1,R2,D,THETAC,THETAM,DELTH
3334 FORMAT(' R1 = ',F6.4,'     R2 = ',F6.4,
+ '     D = ',F6.3,' THETAC = ',F6.3,
+ '     THETAM = ',F6.3,'     DELTH = ',F6.4)
C -----
C           ZERO INTEGRATION SUMS AND INTEGRATE TERMS
C           INVOLVING A(THETA)*POWERS OF COS(THETA) AND SIN(THETA)
C           FROM THETA=0 TO THETA=THETAM
C -----
SMS=0.
SMCS=0.
SMC2S=0.
SMS3=0.
SMC3S=0.
SMCS3=0.
THETA=0.
10  THETA=THETA+DELTH
    IF(THETA.GT.THETAM) GOTO 20
    ATH=A(THETA)
    SINTH=SIN(THETA)
    COSTH=COS(THETA)
    SMS=SMS+ATH*SINTH
    SMCS=SMCS+ATH*COSTH*SINTH
    SMC2S=SMC2S+ATH*COSTH**2*SINTH
    SMS3=SMS3+ATH*SINTH**3
    SMC3S=SMC3S+ATH*COSTH**3*SINTH
    SMCS3=SMCS3+ATH*COSTH*SINTH**3
    GOTO 10
```

```
//ANNIE      JOB      (98221,CZG,SRL), 'N. GEHRELS', TIME=(2,00)
//      EXEC FORTG
//FORT      DD      *
C -----
C           READ IN DETECTOR RADII AND SEPARATION
C -----
C           DIMENSION LABEL(19)
COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
5  READ(5,1111,END=999) LABEL
1111 FORMAT(19A4)
      READ(5,1112) R1,R2,D
1112 FORMAT(3F10.0)
C -----
C           INITIALIZE VARIABLES
C -----
CALL INIT(R1,R2,D)
DELTH=0.002
C -----
C           OUTPUT THE A HEADER
C -----
WRITE(6,3333) LABEL
3333 FORMAT(////1X19A4)
WRITE(6,3334) R1,R2,D,THETAC,THETAM,DELTH
3334 FORMAT(' R1 = ',F6.4,' R2 = ',F6.4,
+ ' D = ',F6.3,' THETAC = ',F6.3,
+ ' THETAM = ',F6.3,' DELTH = ',F6.4)
C -----
C           ZERO INTEGRATION SUMS AND INTEGRATE TERMS
C           INVOLVING A(THETA)*POWERS OF COS(THETA) AND SIN(THETA)
C           FROM THETA=0 TO THETA=THETAM
C -----
SMS=0.
SMCS=0.
SMC2S=0.
SMS3=0.
SMC3S=0.
SMCS3=0.
THETA=0.
10 THETA=THETA+DELTH
IF(THETA.GT.THETAM) GOTO 20
ATH=A(THETA)
SINTH=SIN(THETA)
COSTH=COS(THETA)
SMS=SMS+ATH*SINTH
SMCS=SMCS+ATH*COSTH*SINTH
SMC2S=SMC2S+ATH*COSTH**2*SINTH
SMS3=SMS3+ATH*SINTH**3
SMC3S=SMC3S+ATH*COSTH**3*SINTH
SMCS3=SMCS3+ATH*COSTH*SINTH**3
GOTO 10
```

```
C -----
C           SCALE BY DELTH
C -----
20   SMS=SMS*DELTH
     SMCS=SMCS*DELTH
     SMC2S=SMC2S*DELTH
     SMS3=SMS3*DELTH
     SMC3S=SMC3S*DELTH
     SMCS3=SMCS3*DELTH
     WRITE(6,4444) SMS,SMCS,SMC2S,SMS3,SMC3S,SMCS3
4444  FORMAT(//' INTEGRAL(A SIN)=',F7.5,10X'INTEGRAL(A COS SIN)=',
+      F7.5/' INTEGRAL(A COS**2 SIN)=',F7.5,
+      ' INTEGRAL(A SIN**3)=',F7.5/' INTEGRAL(A COS**3 SIN)=',
+      F7.5,' INTEGRAL(A COS SIN**3)=',F7.5)
C -----
C           CALCULATE THE VARIOUS TERMS
C -----
      TRMK01=2.*PI*SMS
      TRMK11=2.*PI*SMCS
      TRMK21=(3.*PI/2.)*(2.*SMC2S-SMS3)
      TRMK22=PI*(3.*SMS3/2.-SMS)
      TRMK31=5.*PI*(2.*SMC3S-3.*SMCS3)/2.
      TRMK32=PI*(15.*SMCS3/2.-3.*SMCS)
C -----
C           OUTPUT THE TERMS
C -----
      WRITE(6,5555) TRMK01,TRMK11,TRMK21,TRMK22,TRMK31,TRMK32
5555  FORMAT(///' K0 = A OMEGA =',FB.4/' K1 =    COS(ALPHA1) *',
+      FB.4/' K2 =    COS**2(ALPHA2) *',FB.4,' + ',FB.4/
+      ' K3 =    COS**3(ALPHA3) *',FB.4,' +    COS(ALPHA3) *',
+      FB.4///)
      GOTO 5
999  STOP
      END
      SUBROUTINE INIT(R1,R2,D)
      COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
C -----
C           THIS SUBROUTINE INITIALIZES VARIABLE THAT ARE NEEDED
C           TO CALCULATED A(THETA)
C -----
      PI=3.14159
      RS=A MIN1(R1,R2)
      ABSR12=ABS(R1-R2)
      THETAC=ATAN(ABSR12/D)
      THETAM=ATAN((R1+R2)/D)
      R1SQR=R1*R1
      R2SQR=R2*R2
      RSSQR=RS*RS
      DSQR=D*D
      DR12=2*D*R1
      DR22=2*D*R2
      RETURN
      END
      FUNCTION A(THETA)
      COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
C -----
C           THIS FUNCTION CALCULATES THE PROJECTED OVERLAP AREA
C           OF THE TWO DETECTORS AS A FUNCTION OF THETA
```

C -----
IF(THETA.GE.THETAC) GOTO 10
A=PI*RSSGR*COS(THETA)
RETURN
10 IF(THETA.LT.THETAM) GOTO 20
A=0.
RETURN
20 TANTH=TAN(THETA)
PHI12=2.*ARCOS((R1SQR+DSQR*TANTH**2-R2SQR)/(DR12*TANTH))
PHI22=2.*ARCOS((R2SQR+DSQR*TANTH**2-R1SQR)/(DR22*TANTH))
A=COS(THETA)*(R1SQR*(PHI12-SIN(PHI12))/2.+
+ R2SQR*(PHI22-SIN(PHI22))/2.)
RETURN
END
//DATA DD *
 NOMINAL L1-L2
0.95 0.95 4.08
 NOMINAL WINDOW-L1
1.74 0.95 1.55
 VGR 1 LET A L1-L2
0.9505 0.9484 4.076
 VGR 1 LET B L1-L2
0.9479 0.9519 4.081
 VGR 1 LET C L1-L2
0.9487 0.9478 4.079
 VGR 1 LET D L1-L2
0.9460 0.9466 4.075
 VGR 2 LET A L1-L2
0.9471 0.9491 4.075
 VGR 2 LET B L1-L2
0.9451 0.9483 4.064
 VGR 2 LET C L1-L2
0.9502 0.9424 4.084
 VGR 2 LET D L1-L2
0.9499 0.9447 4.061
//
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